# Math III Unit 4: RATIONAL EXPRESSIONS AND EQUATIONS

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**Main topics of instruction:**
1) Direct, Joint, and Inverse Variation
2) Rational Functions and their Graphs
3) Multiplying and Dividing Rational Expressions
4) Adding and Subtracting Rational Expressions
5) Solving Rational Equations and Inequalities

## Day 1: Direct, Joint, and Inverse Variation

**Direct Variation:** A relationship between two variables such that one variable increases or decreases proportionally with the other.

**Identifying Direct Variation**

$y=kx$

$\frac{y}{x}=k$

Does y vary directly with x?

$k=\frac{y}{x}$

| x | y |
|---|---|
| 2 | 8 |
| 3 | 12 |
| 5 | 20 |

$\frac{8}{2}=4$

$\frac{12}{3}=4$

$\frac{20}{5}=4$

yes! k is always 4.

| x | y |
|---|---|
| -1 | -2 |
| 3 | 4 |
| 6 | 7 |

$\frac{-2}{-1}=2$

$\frac{4}{3}=1.\overline{3}$

no! k is not constant

What about in equations? Central question: Can you get the equation in the form $y=kx$?

**Example 1:** $3y = 2x$

$y=\frac{2}{3}x$ yes!

**You try!** Can you get $y = 2x + 3$ into $y = kx$ form? No!

Make a table of values, and use it to prove your answer.

$k=\frac{y}{x}$

| x | y |
|---|---|
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |

$\frac{5}{1}=5$

$\frac{7}{2}=3.5$

$y=2x+3$ k isn't constant!

What one rule have you learned from this example? If your equation has a y-intercept, direct variation cannot exist!

y ↓ ↙x

**Example 2:** A dripping faucet wastes 1 cup of water if it drips for 3 minutes. The amount of water wasted varies directly with the amount of time passed. Write an equation of direct variation.

$1 = k(3)$

$\frac{1}{3} = k$

$\boxed{y = \frac{1}{3}x}$

**You try!** The circumference of a circle varies directly with the diameter. What is the constant of variation? Find the diameter of a circle with circumference 105 cm.

$k = \pi$

$105 = \pi d$

$\boxed{d = 33.42 \text{ cm}}$

$C = \pi d$

↑ this is k!

**Example 3:** Write and equation of direct variation that passes through (9, -1).

x y

$-1 = k(9)$

$-\frac{1}{9} = k$

$\boxed{y = -\frac{1}{9}x}$

**You try!** Write an equation of direct variation that passes through (-3, 14).

$14 = k(-3)$

$-\frac{14}{3} = k$

$\boxed{y = -\frac{14}{3}x}$

**Example 4:** y varies directly with x, and x = 27 when y = 51. Find x when y = -17.

$51 = k(27)$

$\frac{51}{27} = k$

$\frac{17}{9} = k$

$y = \frac{17}{9}x$

$-17 = \frac{17}{9}(x)$

$\boxed{x = -9}$

**You try!** y varies directly with x. If x = 1 when y = 5, find y when x = 3.

$5 = k(1)$

$5 = k$

$y = 5x$

$y = 5(3)$

$\boxed{y = 15}$

**Inverse Variation:** A relationship between variables such that when one increases, the other decreases (and vice versa) proportionally.

**Modeling Inverse Variation**

$y = \frac{k}{x}$, $xy = k$

**Example 5:** x and y vary inversely. x = 3 when y = -5. Write the function of inverse variation.

$-5 = \frac{k}{3}$

$-15 = k$

$\boxed{y = \frac{-15}{x}}$

**You try!** Decide which type of variation is represented by the data: direct, inverse, or neither.

direct

| | | | | |
|---|---|---|---|---|
| ↑ | x | 0.5 | 2 | 6 |
| ↑ | y | 1.5 | 6 | 18 |

$\frac{1.5}{.5}=3 \quad \frac{6}{2}=3 \quad \frac{18}{6}=3 \qquad k=\frac{y}{x} \quad k=xy$

inverse

| | | | | |
|---|---|---|---|---|
| ↑ | x | 0.2 | 0.6 | 1.2 |
| ↓ | y | 12 | 4 | 2 |

$12\cdot0.2=2.4 \quad 4\cdot0.6=2.4 \quad 2\cdot1.2=2.4$

| | | | | |
|---|---|---|---|---|
| ↑ | x | 1 | 2 | 3 |
| ↓ | y | 2 | 1 | 0.5 |

neither

$1\cdot2=2 \quad 2\cdot1=2 \quad 3\cdot0.5=1.5$

## Joint Variation

direct: $y=kx$
inverse: $y=\frac{k}{x}$

| Description | Equation |
|---|---|
| y varies directly with the square of x. | $y=kx^2$ |
| y varies inversely with the cube of x. | $y=\frac{k}{x^3}$ |
| z varies jointly with x and y. joint = direct w/ more variables! | $z=kxy$ |
| z varies jointly with x and y and inversely with w. | $z=\frac{kxy}{w}$ |
| z varies directly with x and inversely with the product of w and y. | $z=\frac{kx}{wy}$ |

**Application:** The volume of a regular tetrahedron varies directly with the cube of the length of an edge. The volume of a regular tetrahedron with edge length 3 is $\frac{9\sqrt{2}}{4}$. Find the formula for the volume of a regular tetrahedron.

$$V=k\ell^3$$

$$\frac{9\sqrt{2}}{4}=k(3)^3$$

$$\frac{9\sqrt{2}}{4}=k(27)$$

$$\frac{9\sqrt{2}}{4\cdot27}=k$$

$$\frac{9\sqrt{2}}{108}=k$$

$$\frac{\sqrt{2}}{12}=k$$

$$\boxed{V=\frac{\sqrt{2}}{12}\ell^3}$$

**Day 2: Rational Functions and Their Graphs**

$$f(x) = \frac{P(x)}{Q(x)}$$

**<u>Points of discontinuity – 2 types</u>**

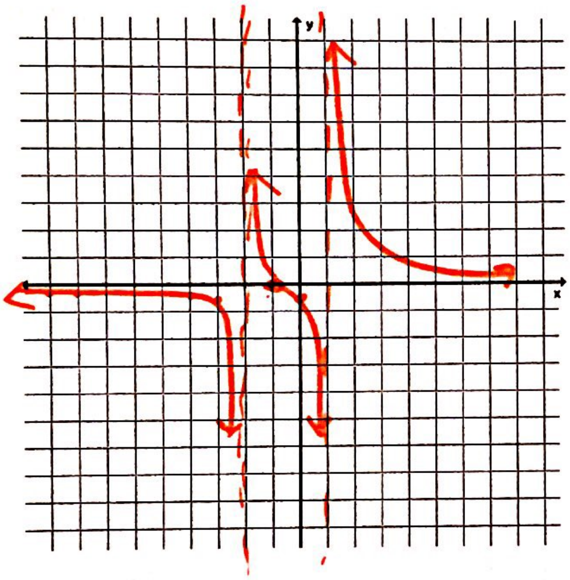
where the denom. = 0.

- **Asymptote:** a line that a function gets closer and closer to, but never touches.

where the num. and denom. = 0.

- **Hole:** a point on the function that the function skips over.

**Example 1:** Let's visualize it! Graph $\frac{(x+1)}{(x-1)(x+2)}$



What happened at x = -2 and x = -1?

asymptotes!

Try plugging in x = 1 to the equation:

$$\frac{(1+1)}{(1-1)(1+2)} = \frac{2}{(0)(3)} = \frac{2}{0} = \text{undef.}$$

Try plugging in x = -2 to the equation:

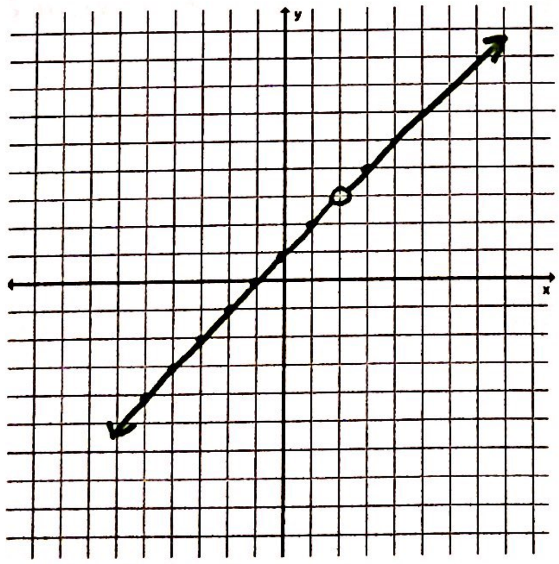
$$\frac{(-2+1)}{(-2-1)(-2+2)} = \frac{(-1)}{(-3)(0)} = \frac{-1}{0} = \text{undef.}$$

The denominator can never equal 0!

Domain: $(-\infty,-2) \cup (-2,1) \cup (1,\infty)$

Range: $(-\infty,\infty)$

**Example 2:** Let's visualize it! Graph $\frac{(x-2)(x+1)}{(x-2)}$



What happened at x = 2?

hole

Try plugging in x = 2 to the equation:

$$\frac{(2-2)(2+1)}{(2-2)} = \frac{0(3)}{0} = \frac{0}{0}$$

both num. & denom. are zero! (makes a hole)

Why do you think a hole was created instead of an asymptote?

both the top and bottom equal 0.

Domain: $(-\infty,2) \cup (2,\infty)$

Range: $(-\infty,3) \cup (3,\infty)$

Ask yourself two questions:

1) What numbers will make the denominator equal 0?
2) Do they make a hole or an asymptote? (or, will they also make the numerator equal 0?)

**Example 3:** What are the points of discontinuity of $y = \frac{1}{x^2+2x+1}$? $= \frac{1}{(x+1)(x+1)}$

We need to factor the bottom to find out what makes it equal 0!

$x^2+2x+1$

$(x^2+x)(+x+1)$

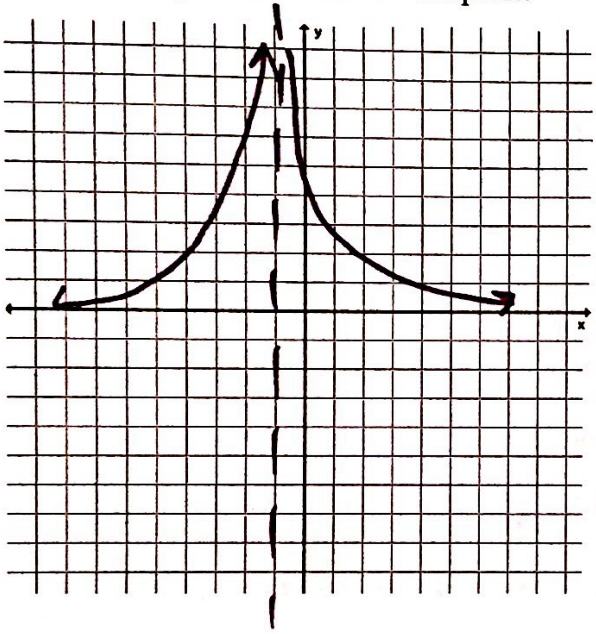
$x(x+1)+1(x+1)$

$(x+1)(x+1)$

$\boxed{x \neq -1}$

asymptote!

Is it an asymptote or a hole? Graph it!



**You try!** Find the points of discontinuity for:

purple

a) $y = \frac{1}{x^2-16}$

$y = \frac{1}{(x+4)(x-4)}$

$x \neq -4, 4$

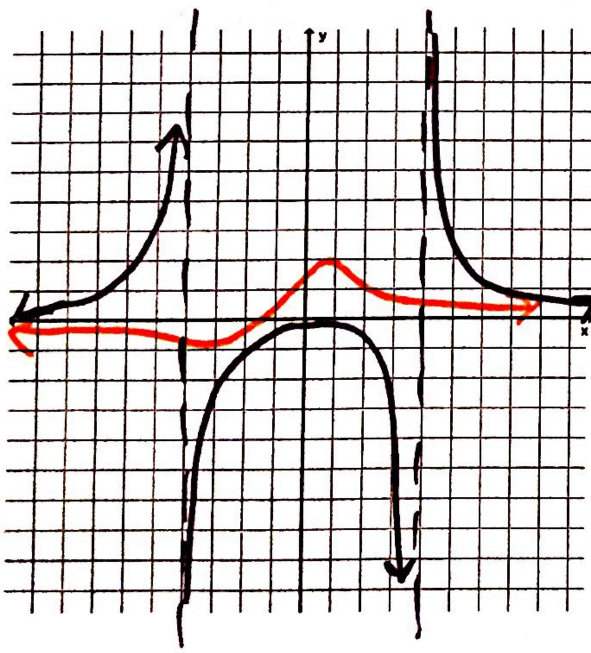
asymptotes

orange

b) $y = \frac{x+1}{x^2+3}$

what would make the denom. equal 0?

nothing!



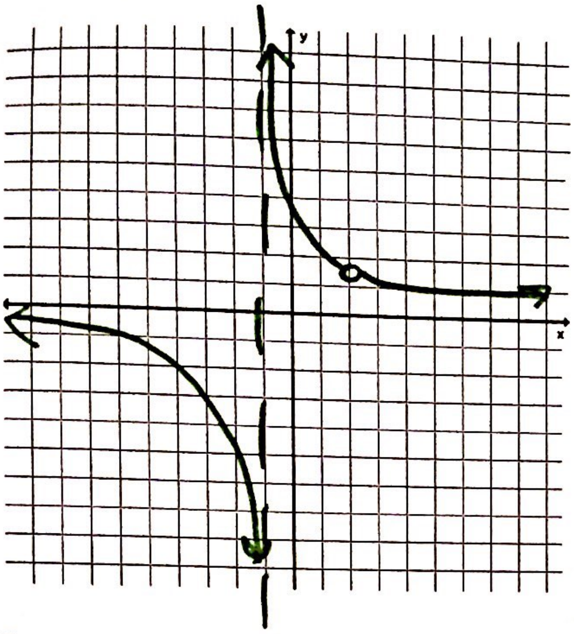
or $y = \frac{(x+3)(x-2)}{(x-2)(x+1)}$. Are they holes or asymptotes?

what makes denom. equal 0?

next page!

$x \neq 2$, $x \neq -1 \rightarrow$ asymp.

↳ also makes num. $=0$! (hole)



## Day 3: More Rational Function Graphs: Horizontal Asymptotes

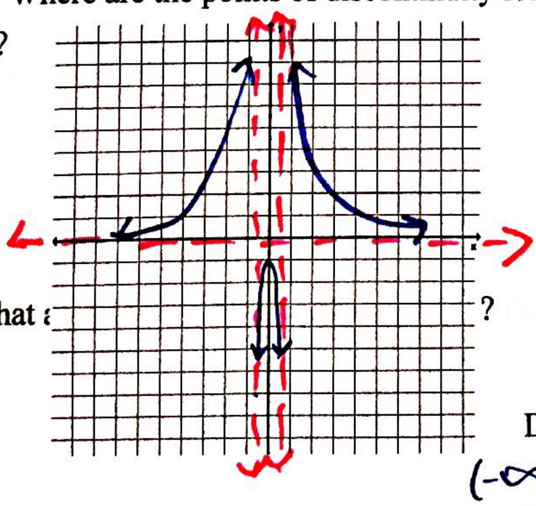
Yesterday, we talked about vertical asymptotes and holes.

But, some graphs also have horizontal asymptotes.

**<u>Rules for Horizontal Asymptotes</u>**

1. **Top Heavy:** There is a horizontal asymptote at none! if the degree on top is higher than the degree on the bottom.
2. **Bottom Heavy:** There is a horizontal asymptote at 0 if the degree on the bottom is higher than the degree on top.
3. **Balanced:** There is a horizontal asymptote at $y = \frac{a}{b}$ if a is the numerator's leading coefficient and b is the denominator's leading coefficient.

**Example 1:** Where are the points of discontinuity for $y = \frac{x+2}{2x^2-4}$, including all asymptotes and/or holes?



$$y = \frac{x+2}{2(x^2-2)}$$

$$y = \frac{x+2}{2(x+\sqrt{2})(x-\sqrt{2})}$$

vertical asymptotes: $x = \sqrt{2}, -\sqrt{2}$

horizontal asymptotes: $y = 0$

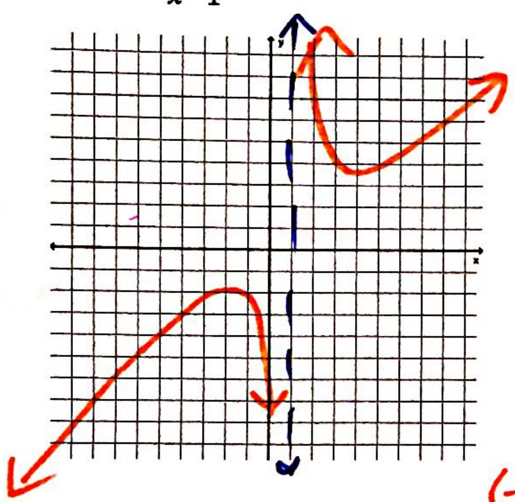
holes: none

**You try!** What ?

Domain: $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

a) $y = \frac{x^2+4}{x-1}$

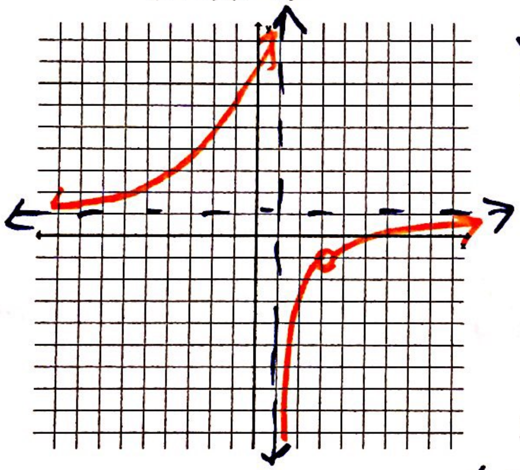


vertical asymptotes: x=1
horizontal asymptotes: none
holes: none

Domain: $(-\infty,1)\cup(1,\infty)$

Range: $(-\infty,-2.47]\cup[6.47,\infty)$

b) $y = \frac{(x-2)(x+3)}{(x+3)(x-1)}$



vertical asymptotes: x=1
horizontal asymptotes: $y=\frac{1}{1}=1$
holes: x=-3

Domain: $(-\infty,-1)\cup(1,3)\cup(3,\infty)$

Range: $(-\infty,0.5)\cup(0.5,1)\cup(1,\infty)$

## Day 4: Multiplying and Dividing Rational Expressions

There is one goal when it comes to multiplying and dividing rational expressions, and that is to

______________________________________________________________.

**Example 1:** Simplify $\frac{x^2+10x+25}{x^2+9x+20}$

What are the restrictions on x? (What would make the denominator = 0?) ____________________